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Solitary Wave Solutions of the Benjamin-Bona-Mahoney-Burgers Equation with Dual Power-Law Nonlinearity

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Abstract: In this paper, we employ the extended tanh function method to find the exact traveling wave solutions and solitary wave solutions involving parameters of the Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity. When these parameters are taken to be special values, the solitary wave solutions are derived from the exact traveling wave solutions. These studies reveal that the Benjamin-Bona-Mahoney-Burgers Equation with Dual Power-law nonlinearity has a rich variety of solutions

Keywords: The extended tanh function method; The Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity; Traveling wave solutions; Solitary wave solutions.

1 Introduction

It is well known that many models in mathematics and physics are described by nonlinear partial differential equations (NLPDEs). The theory of solitons has contributed to understanding many experiments and complex phenomena in mathematical physics. Thus, it is of interest to evaluate new solitary wave solutions of these equations. So that, during the past five decades, a lot of method was discovered by a diverse group of scientists to solve the nonlinear partial differential equations. for example, the modified simple equation method [1, 2, 3, 4], $\left(\frac{G'}{G}\right)$ -expansion method [5, 6, 7], modified $\left(\frac{G'}{G}\right)$ -expansion method [8, 9], extended Jacobi elliptic function method [10, 11], $\exp -\varphi(\xi)$ -expansion method [12, 13, 14], extended $\exp -\varphi(\xi)$ -expansion method [15], Riccati-Bernoulli Sub-ODE method [16, 17], the extended tanh- fuction method [18, 19, 20, 21, 22] and so on. In this paper, we shall use the extended tanh function method to find the exact and solitary wave solutions of Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity [23]. This equation is an

alternative to the Korteweg-de Vries (KdV) equation, and describes the unidi-rectional propagation of small-amplitude long waves on the surface of the water in a channel. The BBM equation is not only convenient for shallow water waves but also for hydromagnetic waves, acoustic waves, and therefore it has more advantages compared with the KdV equation. The main idea of this method is finding the exact solutions of any models which can be expressed by a polynomial of $\phi(\xi)$ which satisfies the nonlinear ordinary differential equations $\phi' = b + \phi^2(\xi)$ where $\xi = x - ct$ while a, c are arbitrary constants to be determined later. The degree of the polynomial can be calculated by the homogenous balance between the highest order derivatives and the nonlinear terms. Equating the coefficients of the same power of $\phi(\xi)$ we get the system of algebraic equation. The constants of the polynomial can be determined by solving the system of algebraic equation by any computer program.

The rest of this paper is arranged as follows: Description of the method (the extended tanh function method) in Section 2. We use this method to find the exact solutions of the nonlinear evolution equations pointed out above

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and also we make the comparison between our results and another result that obtained by three different method in Section 3. Conclusions are given in Section 4.

2 Basic steps of the modified extended tanh-function method

Consider the following nonlinear evolution equation

$$F(u, u_t, u_x, u_{tt}, u_{xx}, \dots) = 0, \quad (1)$$

where F is a polynomial in $u(x, t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method :

Step 1. We use the wave transformation

$$u(x, t) = u(\xi), \quad \xi = x - ct, \quad (2)$$

where c is a constant, to reduce Eq.(1) to the following ODE:

$$P(u, u', u'', u''', \dots) = 0, \quad (3)$$

where P is a polynomial in $u(\xi)$ and its total derivatives, while $\left\{ ' = \frac{d}{d\xi} \right\}$.

Step 2. Suppose the solution of Eq.(3) has the form:

$$u(\xi) = a_0 + \sum_{i=1}^N (a_i \phi^i + b_i \phi^{-i}), \quad (4)$$

where a_i, b_i are constants to be determined, such that $a_N \neq 0$ or $b_N \neq 0$ and ϕ satisfies the Riccati equation

$$\phi' = b + \phi^2, \quad (5)$$

Step 3. Determine the positive integer N in Eq.(4) by balancing the highest order derivatives and the nonlinear terms.

Step 4. Substitute Eq.(4) along Eq.(5) into Eq.(3) and collecting all the terms of the same power ϕ^i , $i = N, N-1, \dots, 1-N, -N$ and equating them to zero, we obtain a system of algebraic equations, which can be solved by Maple or Mathematica to get the values of a_i and b_i .

Step 5. Substituting these values and the solutions of Eq.(5) into Eq.(4) we obtain the exact solutions of Eq.(1).

3 The Benjamin-Bona-Mahoney-Burgers Equation with Dual Power-Law Nonlinearity

Consider the Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity [23]

$$u_t + au_x + (b_2 u^{2n} + b_3 u) u_x + cu_{xx} + ku_{xxt} = 0, \quad (6)$$

Where a represents the strength of a defection or drifting, b_2, b_3 measure the strength of the two nonlinear terms

with the exponent n being the power law nonlinearity parameter while the parameters c, k are the dissipative diffraction coefficient. When $(n = 1)$, Eq. (6) can be written as the following:

$$u_t + au_x + (b_2 u^2 + b_3 u) u_x + cu_{xx} + ku_{xxt} = 0, \quad (7)$$

using the transformation $u(\xi) = u(x, t)$ where $(\xi = x - ct)$, we get

$$(a - c_1) u' + (b_2 u^2 + b_3 u) u' + cu'' - kc_1 u''' = 0. \quad (8)$$

Balancing $u^2 u'$ and u''' we obtain $(2M + M + 1 = M + 3) \Rightarrow (M = 1)$, so that the solution of Eq. (8) can be written in the form:

$$u(\xi) = a_0 + a_1 \phi + \frac{b_1}{\phi}. \quad (9)$$

Substituting Eq. (9) and its derivatives into Eq. (8) and collecting all term with the same power of (ϕ^i) where $(i = 4, 3, 2, 1, 0, -1, -2, -3, -4)$ and equating them to zero, we obtain the system of algebraic equations. Solving this system of equation, we obtain:

Case 1.

$$a = \frac{16bkb_2^2b_3^4 - 12b^4c^2k + 3b^2kb_3^4 + 2b_2^2b_3^4}{12kb^2b_2b_3^2},$$

$$a_0 = \frac{-b_3^2 + 2bc}{2b_3b_2}, a_1 = \frac{-b_3}{b}, b_1 = -ba_1, c_1 = \frac{b_2b_3^2}{6kb^2}.$$

So that, the exact traveling wave solutions of Eq. (8) will be in the form

$$u(\xi) = \frac{-b_3^2 + 2bc}{2b_3b_2} - \frac{b_3}{b} \phi - \frac{ba_1}{\phi}. \quad (10)$$

The solitary wave solutions will be in the form:

When $(b < 0)$, we get:

$$u(\xi) = \frac{-b_3^2 + 2bc}{2b_3b_2} + \frac{b_3\sqrt{-b}}{b} \tanh(\sqrt{-b}\xi) \quad (11)$$

$$-a_1\sqrt{-b} \coth(\sqrt{-b}\xi), \quad (12)$$

or

$$u(\xi) = \frac{-b_3^2 + 2bc}{2b_3b_2} + \frac{b_3\sqrt{-b}}{b} \coth(\sqrt{-b}\xi) \quad (13)$$

$$-a_1\sqrt{-b} \tanh(\sqrt{-b}\xi). \quad (14)$$

When $(b > 0)$, we get:

$$u(\xi) = \frac{-b_3^2 + 2bc}{2b_3b_2} - \frac{b_3\sqrt{b}}{b} \tan(\sqrt{b}\xi) - a_1\sqrt{b} \cot(\sqrt{b}\xi), \quad (15)$$

or

$$u(\xi) = \frac{-b_3^2 + 2bc}{2b_3b_2} - \frac{b_3\sqrt{b}}{b} \cot(\sqrt{b}\xi) - a_1\sqrt{b} \tan(\sqrt{b}\xi), \quad (16)$$

When $(b = 0)$, we get:

The trivial solution.

Case 2.

$$a = \frac{4bkb_2^2b_3^4 - 12b^4c^2k + 3b^2kb_3^4 + 2b_2^2b_3^4}{12kb^2b_2b_3^2},$$

$$a_0 = \frac{-b_3^2 + 2bc}{2b_3b_2}, a_1 = 0, b_1 = \pm \sqrt{\frac{6ckb^2}{b_2}}, c_1 = \frac{b_2b_3^2}{6kb^2}.$$

So that, the exact traveling wave solutions of Eq. (8) will be in the form

$$u(\xi) = \frac{-b_3^2 + 2bc}{2b_3b_2} \pm \sqrt{\frac{6ckb^2}{b_2}} \frac{1}{\phi}. \quad (17)$$

The solitary wave solutions will be in the form:

When ($b < 0$), we get:

$$u(\xi) = \frac{-b_3^2 + 2bc}{2b_3b_2} \mp \sqrt{\frac{-6ckb}{b_2}} \coth(\sqrt{-b}\xi), \quad (18)$$

or

$$u(\xi) = \frac{-b_3^2 + 2bc}{2b_3b_2} \mp \sqrt{\frac{-6ckb}{b_2}} \tanh(\sqrt{-b}\xi). \quad (19)$$

When ($b > 0$), we get:

$$u(\xi) = \frac{-b_3^2 + 2bc}{2b_3b_2} \pm \sqrt{\frac{6ckb}{b_2}} \cot(\sqrt{b}\xi), \quad (20)$$

or

$$u(\xi) = \frac{-b_3^2 + 2bc}{2b_3b_2} \pm \sqrt{\frac{6ckb}{b_2}} \tan(\sqrt{b}\xi). \quad (21)$$

When ($b = 0$), we get:

$$u(\xi) = \frac{-b_3^2}{2b_3b_2}. \quad (22)$$

Case 3.

$$a = \frac{2c^2b_2(2bk + 1)}{3kb_3^2}, a_0 = a_1 = 0,$$

$$b_1 = \pm b \sqrt{\frac{6c_1k}{b_2}}, c_1 = \frac{2c^2b_2}{3kb_3^2}.$$

So that, the exact traveling wave solutions of Eq. (8) will be in the form

$$u(\xi) = \pm b \sqrt{\frac{6c_1k}{b_2}} \frac{1}{\phi}. \quad (23)$$

The solitary wave solutions will be in the form:

When ($b < 0$), we get:

$$u(\xi) = \mp \sqrt{\frac{-6c_1kb}{b_2}} \coth(\sqrt{-b}\xi), \quad (24)$$

or

$$u(\xi) = \mp \sqrt{\frac{-6c_1kb}{b_2}} \tanh(\sqrt{-b}\xi). \quad (25)$$

When ($b > 0$), we get:

$$u(\xi) = \pm \sqrt{\frac{-6c_1kb}{b_2}} \cot(\sqrt{b}\xi), \quad (26)$$

or

$$u(\xi) = u(\xi) = \pm \sqrt{\frac{-6c_1kb}{b_2}} \tan(\sqrt{b}\xi). \quad (27)$$

When ($b = 0$), we get:

The trivial solution.

Remark:

All the obtained results have been checked with Maple 16 by putting them back into the original equation and found correct.

Comparison:

In this research, a good numerous comparison between our results and that obtained by another researchers in [23]. they used a several methods to obtain exact and solitary wave solutions for the Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity.

We obtain from our comparison:

Solutions in case 1.

Eq.(11) is similar to Eq. (31) in [23] when ($\Delta = \frac{c(-b_3^2 + 2bc)}{a_1b_2b_3}, b = \frac{-\Delta^2}{16}, a_1 = \frac{-b_3}{b}, \lambda = c$). While Eqs.(13), (15), (16) are new form of solution for Eq.(8).

Solutions in case 2.

Eqs.(18), (19) is similar to Eqs. (27), (28) in [23] when ($\Delta = 2\sqrt{-b}, a_1^2 = \frac{6c^3k}{b_2}, c = \frac{2b_3b_2\sqrt{-b}}{-b_3^2 + 2bc}$). While Eqs.(20), (21), (22) are new form of solution for Eq.(8).

Solutions in case 3.

Eqs.(24), (25) is similar to Eqs. (48), (49) in [23] when ($n = 1, b = \frac{-v^2}{2}, c_1 = \frac{2(a-vc)}{v^2(1+kv^2)}, \xi_0 = 0$). While Eqs.(26), (27) are new form of solution for Eq.(8).

4 Conclusion

The extended tanh function method has been applied in this paper to find the exact traveling wave solutions and then the solitary wave solutions the Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity. Let us compare our results obtained in the present article with the well-known results obtained by other authors using different methods as follows: Our results the Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity are new and different from those obtained in [23]. The obtained exact solutions can be used as benchmarks against the numerical simulations in theoretical physics and fluid mechanics.

Competing interests

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors. The author did not have any competing interests in this research.

Author's contributions

All parts contained in the research carried out by the researcher through hard work and a review of the various references and contributions in the field of mathematics and the physical Applied.

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